4 Operation of Indices

Mastering the operation of indices is the foundation to addition and subtraction of polynomials.

The same goes for multiplication and division.

In $\underbrace{a \bullet a \bullet a \dots a}_{m \text{ number of } `a\text{'s}} = a^m$, a is called the base and m is the power, or index, or exponent.

Some properties of indices:

If a, m and n are all positive integers, then

1. $a^m \bullet a^n = a^{m+n}$
2. $(a^m)^n = a^{mn}$
3. $(ab)^n = a^n b^n$
4. $a^m \div a^n = a^{m-n}$ for $a \neq 0$

When $a \neq 0$, $a^n \div a^n = a^{n-n}$

$$= a^0$$

$$= 1$$

Also, $a^{-n} = a^{0-n}$

$$= \frac{a^0}{a^n}$$

$$= \frac{1}{a^n}$$

Example 1

Simplify each of the following:

(a) $x^3 \cdot x^4$

(b) $a^5 \cdot a^4$

(c) $(m + n)^3 \cdot (m + n)^2$

(d) $x^{n+1} \cdot x^{2n+3}$

(e) $(3m^2n^3)^3$

Solution:

(a) $x^3 \cdot x^4 = x^3 + 4$
$= \boldsymbol{x^7}$

(b) $a^5 \cdot a^4 = a^{5+4}$
$= \boldsymbol{a^9}$

(c) $(m + n)^3 \cdot (m + n)^2 = (m + n)^{3+2}$
$= \boldsymbol{(m + n)^5}$

(d) $x^{n+1} \cdot x^{2n+3} = x^{n+1+2n+3}$
$= \boldsymbol{x^{3n+4}}$

(e) $(3m^2n^3)^3 = 3^3m^{2 \times 3}n^{3 \times 3}$
$= \boldsymbol{27m^6n^9}$

Example 2

Solve for the value of the unknown in each of the following:

(a) $3^x = 243$

(b) $5^m = 625$

(c) $16^x \cdot 5^{2x} = 400$

(d) $2^{2n+2} + 2^{2n} = 160$

Solution:

(a) $3^x = 243 = 81 \times 3$
$= 3 \times 3 \times 3 \times 3 \times 3$
$= 3^5$
$x = \mathbf{5}$

(b) $5^m = 625 = 5 \times 5 \times 5 \times 5$
$= 5^4$
$m = \mathbf{4}$

(c) $16^x \cdot 5^{2x} = 4^{2x} \cdot 5^{2x}$
$= (20)^{2x} = 400 = (20)^2$
$2x = 2$
$x = \mathbf{1}$

(d) $2^{2n+2} + 2^{2n} = 2^{2n} \cdot (2^2 + 1)$
$= 2^{2n} \cdot 5$
$160 = 32 \cdot 5$
$= 2^5 \cdot 5$
$2n = 5$
$n = \mathbf{2.5}$

Example 3

Rank 2^{777}, 3^{555} and 4^{444} in ascending order in terms of their values.

Solution:

$$\begin{aligned} 2^{777} &= (2^7)^{111} \\ &= (128)^{111} \\ 3^{555} &= (3^5)^{111} \\ &= (243)^{111} \\ 4^{444} &= (4^4)^{111} \\ &= (256)^{111} \end{aligned}$$

$$\mathbf{4^{444} > 3^{555} > 2^{777}}$$

Example 4

Given $a^m = \frac{1}{3}$ and $a^n = 6$, find the value of a^{2m+3n}.

Solution:

$$\begin{aligned} a^{2m+3n} &= (a^m)^2 \cdot (a^n)^3 \\ &= \left(\frac{1}{3}\right)^2 \cdot (6)^3 \\ &= \frac{1}{9} \cdot 216 \\ &= \mathbf{24} \end{aligned}$$

Example 5

Given $3x + 5y = 2$, find the value of $27^x \cdot 243^y$.

Solution:

$$\begin{aligned} 27^x &= 3^{3x} \\ 243^y &= 3^{5y} \\ 27^x \cdot 243^y &= 3^{3x} \cdot 3^{5y} \\ &= 3^{3x+5y} \end{aligned}$$

We know $3x + 5y = 2$.

$$\begin{aligned} 3^{3x+5y} &= 3^2 \\ &= \mathbf{9} \end{aligned}$$

Example 6

Given $m + n = 11$ and $mn = 9$, find the value of $m^2 + n^2$.

Solution:

$$m + n = 11$$

Squaring both sides,

$$(m + n)^2 = 11^2$$
$$m^2 + 2mn + n^2 = 121$$
$$m^2 + n^2 = 121 - 2mn$$
$$= 121 - 2(9)$$
$$= \mathbf{103}$$

Example 7

If $a + b = 4$ and $a^3 + b^3 = 28$, $a^2 + b^2 = ?$

$$\begin{bmatrix} \text{Hint: } (a + b)^2 = a^2 + 2ab + b^2 \\ \qquad (a + b)^3 = (a + b)(a^2 - ab + b^2) \end{bmatrix}$$

Solution:

$$(a + b)^3 = (a + b)(a^2 - ab + b^2)$$

We know $a + b = 4$.

$$(a + b)^3 = 4(a^2 - ab + b^2)$$
$$4(a^2 - ab + b^2) = 28$$
$$a^2 - ab + b^2 = 7$$
$$2a^2 - 2ab + 2b^2 = 14 \quad \text{----(1)}$$

From $a + b = 4$, we have

$$(a + b)^2 = 4^2$$
$$a^2 + 2ab + b^2 = 16 \quad \text{----(2)}$$

(1) + (2),

$$3a^2 + 3b^2 = 30$$
$$a^2 + b^2 = \mathbf{10}$$

Practice

1. What is the ones digit of 7^{25}?

(A) 7
(B) 9
(C) 3
(D) 1
(E) 5 ( )

2. If $2^a = 3$, $2^b = 6$ and $2^c = 12$, which of the following is true?

(A) $a + b = c$
(B) $2b > a + c$
(C) $a + b < c$
(D) $2b = a + c$
(E) uncertain ( )

3. What is the value of n in $3^{2n+1} + 4^n = 259$?

(A) $\frac{1}{2}$
(B) 1
(C) 2
(D) 3
(E) 4 ( )

4. If $(x + y)^2 = 9$ and $(x - y)^2 = 1$, $x^2 + y^2 = ?$

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5 ( )

5. If $2^a \cdot 3^b = 2^c \cdot 3^d = 36$, find the value of $(a - 1)(d + 1) + (b + 1)(c - 1)$.

(A) 4
(B) 5
(C) 6
(D) 7
(E) 8 ( )

6. If $a = 5x^2 - 4x + 2$ and $b = 5x^2 - 4x - 3$, is $a > b$ true?

7. Solve for the value of m in each of the following:
(a) $5^m \cdot 25^{2m} = 125^{2m-1}$
(b) $5^m \cdot 25^m \cdot 125^m \cdot 625^m = 5^{30}$

8. Find the value of a in $10x^3 + ax^2 + 5x + 2$ by expanding $(2x^2 + 1)(5x + 2)$.

9. It is given $3^x = \frac{2}{9}$ and $3^y = 27$. Find the value of 3^{3x+2y}.

10. Suppose $(x + y)^2 = 18$ and $(x - y)^2 = 12$. Find the value of $x^2 + y^2$.

11 If $a^x + a^{-x} = 3$, find the value of $a^{2x} + a^{-2x}$.

$\begin{bmatrix}\text{Hint 1: } a^x \cdot a^{-x} = a^{x-x} = a^0 = 1 \\ \text{Hint 2: } (a+b)^2 = (a+b)\cdot(a+b) = a^2 + 2ab + b^2\end{bmatrix}$

12 If $3x + 5y + 3 = 0$, find the value of $8^{x+2} \cdot 32^y$.

13 Suppose $a + b = 5$ and $a^3 + b^3 = 35$, find the value of $a^2 + b^2$.

[Hint: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$]

14 Given $m^2 + m = \frac{1}{4}$, find the value of $8m^4 + 20m^3 + 13m^2$.